

THE CATEGORY AND OPERAD OF MATCHING DIALGEBRAS

YONG ZHANG, CHENGMING BAI, AND LI GUO

ABSTRACT. This paper gives a systematic study of matching dialgebras corresponding to the operad $As^{(2)}$ in [20] as the only Koszul self dual operad there other than the operads of associative algebras and Poisson algebras. The close relationship of matching dialgebras with semi-homomorphisms and matched pairs of associative algebras are established. By anti-symmetrizing, matching dialgebras are also shown to give compatible Lie algebras, pre-Lie algebras and PostLie algebras. By the rewriting method, the operad of matching dialgebras is shown to be Koszul and the free objects are constructed in terms of tensor algebras. The operadic complex computing the homology of the matching dialgebras is made explicit.

CONTENTS

1. Introduction	1
2. Matching dialgebras, matched pairs and Lie type algebras	2
2.1. Definitions of matching dialgebras	2
2.2. Matching dialgebras and matched pairs of algebras	3
2.3. Matching dialgebras and Lie type algebras	5
3. Koszuality, free objects and operadic homology of the matching dialgebras	6
3.1. The operad $As^{(2)}$ of matching dialgebras is Koszul	7
3.2. Free matching dialgebras	7
3.3. Operadic homology of matching dialgebras	10
References	12

1. INTRODUCTION

The Encyclopedia of Types of Algebras [20] is a collection of algebraic structures (operads) together with their known properties. The operad $As^{(2)}$ is one of the few in the collection that have not been studied systematically. This operad in fact has very good properties. For instance, as well as being Koszul, it is also Koszul self dual, the only such structure in [20] other than the associative algebra and the Poisson algebra. Our motivation of studying this structure, under the name matching dialgebra, came from its close relationship with the concept of a matched pair [3] of two associative algebras, the latter being a natural generalization of semi-direct sums of an associative algebra with its bimodules. In fact, the notion of the matched pair of Lie algebras played an important role in the study of Lie bialgebras [13] and geometric structures on Lie groups, such as complex product structures [1]. Later, a matched pair of associative algebras played an essential role in the study of double constructions of Frobenius algebras and

2000 *Mathematics Subject Classification.* 16Y99, 18D50, 18G60, 17B99.

Key words and phrases. matching dialgebra, matched pairs, semi-direct sum, pre-Lie algebra, PostLie algebra, rewriting method, Koszul operad, free algebra, Quillen homology.

Corresponding author: Li Guo; e-mail: liguo@rutgers.edu; phone: (973) 353-3917; fax: (973) 353-5270.

Connes cocycles, which are equivalent to antisymmetric infinitesimal bialgebras and dendriform D-bialgebras respectively [3].

In this paper we show that the matching dialgebra has many other interesting properties. We also find further relations between matching dialgebras with semi-homomorphisms, the above mentioned matched pairs and some important Lie type algebras: compatible Lie algebras in integrable systems and Yang-Baxter equations [8, 9, 11, 16], pre-Lie algebras in deformations of associative algebras, affine geometry and quantum field theory [5, 6, 19] and PostLie algebras in operads, integrable systems and classical Yang-Baxter equations [4, 18]. We construct the graded-module of the operad of matching dialgebras and the free objects from tensor algebras. We show that its operad is Koszul and construct its homology.

In § 2.1, we give the definition of a matching dialgebra and study its relationship with semi-homomorphisms. The relationship between matching dialgebras and matched pairs of associative algebras is studied in § 2.2. In § 2.3, we derive from matching dialgebras some useful Lie type algebras, such as compatible Lie algebras, pre-Lie algebras and PostLie algebras, in a way similar to the process of deriving Lie algebras from associative algebras by anti-symmetrizing. We apply the rewriting method to prove that the operad $As^{(2)}$ of matching dialgebras is Koszul in § 3.1, and construct the graded space of the operad $As^{(2)}$ and free matching dialgebras in § 3.2. The operadic homology of $As^{(2)}$ -algebras is constructed in § 3.3.

2. MATCHING DIALGEBRAS, MATCHED PAIRS AND LIE TYPE ALGEBRAS

Throughout this paper, algebras and modules are assumed to be over a field \mathbf{k} . The tensor product $\otimes_{\mathbf{k}}$ over \mathbf{k} is simply denoted by \otimes .

2.1. Definitions of matching dialgebras. The following algebra structure is denoted by $As^{(2)}$ -algebra in [20] and got its name from its close relationship with matched pairs of associative algebras. See Section 2.2.

Definition 2.1. (a) A **matching dialgebra (MDA)** is a \mathbf{k} -module A with two binary operations:

$$\circ_1, \circ_2 : A \otimes A \rightarrow A,$$

satisfying the following **MDA axioms**:

- (a) \circ_1 and \circ_2 are associative;
- (b) the following equations hold

$$(1) \quad (x \circ_1 y) \circ_2 z = x \circ_1 (y \circ_2 z), \quad (x \circ_2 y) \circ_1 z = x \circ_2 (y \circ_1 z), \quad \forall x, y, z \in A.$$

- (b) Let (R, \circ_1, \circ_2) and (R', \circ'_1, \circ'_2) be two matching dialgebras. A linear map $f : R \rightarrow R'$ is a **homomorphism of matching dialgebras** if f is a \mathbf{k} -module homomorphism and for all $a, b \in R$:

$$(2) \quad f(a \circ_1 b) = f(a) \circ'_1 f(b) \quad \text{and} \quad f(a \circ_2 b) = f(a) \circ'_2 f(b).$$

We give some examples of matching dialgebras from semi-homomorphisms.

Definition 2.2. Let A be a \mathbf{k} -algebra.

- (a) A linear map $f : A \rightarrow A$ is called a **left semi-homomorphism** (resp. **right semi-homomorphism**) if f satisfies

$$f(xy) = xf(y) \text{ (resp. } f(xy) = f(x)y), \quad \forall x, y \in A.$$

- (b) A linear map $f : A \rightarrow A$ is called a **semi-homomorphism** if f is both a left and a right semi-homomorphism.

Remark 2.3. (a) Let $A = A_1 \oplus A_2$ be a direct sum of ideals. Then the projections p_1 and p_2 of A to A_1 and A_2 are semi-homomorphisms. In this sense, semi-homomorphisms are natural generalizations of projections.

- (b) A left (resp. right) semi-homomorphism is a left (resp. right) Baxter operator, in the sense that f satisfies

$$f(x)f(y) = f(f(x)y) \text{ (resp. } f(x)f(y) = f(xf(y))), \quad \forall x, y \in A.$$

Proposition 2.4. Let (A, \cdot) be a \mathbf{k} -algebra.

- (a) Let f be a left (resp. right) semi-homomorphism. Then the triple (A, \circ_1, \circ_2) with

$$x \circ_1 y := x \cdot y, \quad x \circ_2 y = f(x) \cdot y, \quad (\text{resp. } x \circ_2 y = x \cdot f(y)), \quad \forall x, y \in A,$$

is a matching dialgebra.

- (b) Let f, g be two semi-homomorphisms. Then the triple (A, \circ_1, \circ_2) with

- (3) $x \circ_1 y := f(x) \cdot y (= f(x \cdot y) = x \cdot f(y)), \quad x \circ_2 y := g(x) \cdot y (= g(x \cdot y) = x \cdot g(y)), \quad \forall x, y \in A,$
is a matching dialgebra.

The proof is a direct computation from the definitions.

2.2. Matching dialgebras and matched pairs of algebras. The notion of a matched pair [3] of associative algebras is obtained by generalizing a semi-direct sum of an associative algebra and its bimodule to a direct sum of the underlying vector spaces of two associative algebras. It provides a natural underlying structure for the so-called “double” structures and hence bialgebra theories. In fact, many bialgebraic structures can be interpreted in terms of matched pairs of the corresponding algebras.

Definition 2.5. ([3]) Let (A, \cdot) and (B, \circ) be two associative algebras. Suppose that there are linear maps $l_A, r_A : A \rightarrow \text{gl}(B)$ and $l_B, r_B : B \rightarrow \text{gl}(A)$ such that (B, l_A, r_A) is a bimodule of A and (A, l_B, r_B) is a bimodule of B . If the following equations are also satisfied, then $(A, B, (l_A, r_A), (l_B, r_B))$ is called a **matched pair** of associative algebras:

$$(4) \quad l_A(x)(a \circ b) = l_A(r_B(a)x)b + (l_A(x)a) \circ b,$$

$$(5) \quad r_A(x)(a \circ b) = r_A(l_B(b)x)a + a \circ (r_A(x)b),$$

$$(6) \quad l_B(a)(x \cdot y) = l_B(r_A(x)a)y + (l_B(a)x) \cdot y,$$

$$(7) \quad r_B(a)(x \cdot y) = r_B(l_A(y)a)x + x \cdot (r_B(a)y),$$

$$(8) \quad l_A(l_B(a)x)b + (r_A(x)a) \circ b - r_A(r_B(b)x)a - a \circ (l_A(x)b) = 0,$$

$$(9) \quad l_B(l_A(x)a)y + (r_B(a)x) \cdot y - r_B(r_A(y)a)x - x \cdot (l_B(a)y) = 0,$$

for any $x, y \in A, a, b \in B$.

The close relationship between matching dialgebras and matched pairs of associative algebras is given by the following theorem.

Theorem 2.6. Let $A_{\circ_1} := (A, \circ_1)$ and $A_{\circ_2} := (A, \circ_2)$ be two \mathbf{k} -algebras. Let L_{\circ_i} (resp. R_{\circ_i}) denote the left (resp. right) regular representation of A_{\circ_i} , $i = 1, 2$. Then the following conditions are equivalent:

- (a) (A, \circ_1, \circ_2) is a matching dialgebra.
- (b) $(A_{\circ_1}, A_{\circ_2}, (L_{\circ_1}, 0), (L_{\circ_2}, 0))$ is a matched pair of associative algebras.
- (c) $(A_{\circ_1}, A_{\circ_2}, (0, R_{\circ_1}), (0, R_{\circ_2}))$ is a matched pair of associative algebras.

Proof. (a) \iff (b). It is easily checked that $(A_{\circ_1}, A_{\circ_2}, (L_{\circ_1}, 0), (L_{\circ_2}, 0))$ is a matched pair of associative algebras if and only if the following equations hold:

$$\begin{aligned}
 (10) \quad & L_{\circ_1}(x)(a \circ_2 b) = (L_{\circ_1}(x)a) \circ_2 b, \\
 (11) \quad & L_{\circ_2}(a)(x \circ_1 y) = (L_{\circ_2}(a)x) \circ_1 y, \\
 (12) \quad & L_{\circ_1}(L_{\circ_2}(a)x)b = a \circ_2 (L_{\circ_1}(x)b), \\
 (13) \quad & L_{\circ_2}(L_{\circ_1}(x)a)y = x \circ_1 (L_{\circ_2}(a)y), \quad \forall x, y, a, b \in A.
 \end{aligned}$$

From Eq. (10), we have

$$(x \circ_1 a) \circ_2 b = x \circ_1 (a \circ_2 b).$$

From Eq. (11), we have

$$(a \circ_2 x) \circ_1 y = a \circ_2 (x \circ_1 y).$$

Thus, if $(A_{\circ_1}, A_{\circ_2}, (L_{\circ_1}, 0), (L_{\circ_2}, 0))$ is a matched pair of associative algebras, then (A, \circ_1, \circ_2) is a matching dialgebra. Note that in this case, Eq. (12) is equivalent to Eq. (11) and Eq. (13) is equivalent to Eq. (10). Conversely, if (A, \circ_1, \circ_2) is a matching dialgebra, then Eq. (10)–Eq. (13) hold. Therefore, $(A_{\circ_1}, A_{\circ_2}, (L_{\circ_2}, 0), (L_{\circ_1}, 0))$ is a matched pair of associative algebras.

(a) \iff (c) follows from a similar argument. \square

Proposition 2.7. ([3]) If $(A, B, (l_A, r_A), (l_B, r_B))$ is a matched pair of associative algebras, then there is an associative algebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B . The product is given by

$$(14) \quad (x, a) * (y, b) = (x \cdot y + l_B(a)y + r_B(b)x, a \circ b + l_A(x)b + r_A(y)a), \quad \forall x, y \in A, a, b \in B.$$

Conversely, every associative algebra whose underlying vector space has a decomposition into the direct sum of two subalgebras can be obtained from a matched pair of the two subalgebras.

From Theorem 2.6 and Proposition 2.7 we have the following direct consequence.

Corollary 2.8. Let A be a vector space with two binary operations $\circ_1, \circ_2 : A \otimes A \rightarrow A$. Define two binary operations $\cdot, \circ : (A \oplus A) \otimes (A \oplus A) \rightarrow (A \oplus A)$ by

$$(15) \quad (a, b) \cdot (c, d) = (a \circ_1 c + a \circ_2 d, b \circ_2 d + b \circ_1 c), \quad \forall a, b, c, d \in A,$$

$$(16) \quad (a, b) \circ (c, d) = (a \circ_1 c + b \circ_2 c, b \circ_2 d + a \circ_1 d), \quad \forall a, b, c, d \in A.$$

respectively. Then the following conditions are equivalent:

- (a) (A, \circ_1, \circ_2) is a matching dialgebra.
- (b) $(A \oplus A, \cdot)$ is an associative algebra.
- (c) $(A \oplus A, \circ)$ is an associative algebra.

2.3. Matching dialgebras and Lie type algebras. We now consider the close relationship between the matching dialgebra and several Lie type algebras that have played important roles in mathematics and mathematical physics as indicated in the introduction. We mainly consider the following binary operations.

Definition 2.9. Let (A, \circ_1, \circ_2) be a matching dialgebra. Define

$$(17) \quad [x, y]_{\circ_1} := x \circ_1 y - y \circ_1 x, \quad [x, y]_{\circ_2} := x \circ_2 y - y \circ_2 x,$$

$$(18) \quad [x, y]_{\circ_1, \circ_2} := x \circ_1 y - y \circ_2 x, \quad [x, y]_{\circ_2, \circ_1} := x \circ_2 y - y \circ_1 x.$$

Remark 2.10. (a) Since both (A, \circ_1) and (A, \circ_2) are associative algebras, $(A, [\ , \]_{\circ_1})$ and $(A, [\ , \]_{\circ_2})$ are Lie algebras.

(b) It is obvious that $(A, -[\ , \]_{\circ_2, \circ_1})$ is the opposite algebra of $(A, [\ , \]_{\circ_1, \circ_2})$.

Definition 2.11. (a) ([11, 14, 15, 17]) Two associative algebras (A, \circ_1) and (A, \circ_2) are called **compatible** if, for any $\alpha, \beta \in \mathbf{k}$, the product

$$(19) \quad x \star y = \alpha x \circ_1 y + \beta x \circ_2 y, \quad \forall x, y \in A,$$

defines an associative algebra.

(b) ([8, 9, 11, 16]) Two Lie algebras $(V, [\ , \]_1)$ and $(V, [\ , \]_2)$ are called **compatible** if, for any $\alpha, \beta \in \mathbf{k}$, the product

$$(20) \quad [x, y] = \alpha [x, y]_1 + \beta [x, y]_2, \quad \forall x, y \in V,$$

defines a Lie algebra.

Proposition 2.12. (a) Let (A, \circ_1, \circ_2) be a matching dialgebra. Then (A, \circ_1) and (A, \circ_2) are compatible associative algebras.

(b) Let (A, \circ_1, \circ_2) be a matching dialgebra. Then $(A, [\ , \]_{\circ_1})$ and $(A, [\ , \]_{\circ_2})$ defined by Eq. (17) are compatible Lie algebras.

The proof follows from a direct computation.

We next study the relationship between matching dialgebras and pre-Lie algebras [5, 6, 19].

Lemma 2.13. Let (A, \circ_1, \circ_2) be a matching dialgebra. With the notations in Eqs. (17) and (18), the following equations hold

$$(21) \quad \begin{aligned} & [[x, y]_{\circ_1, \circ_2}, z]_{\circ_1, \circ_2} - [x, [y, z]_{\circ_1, \circ_2}]_{\circ_1, \circ_2} - ([y, x]_{\circ_1, \circ_2}, z]_{\circ_1, \circ_2} - [y, [x, z]_{\circ_1, \circ_2}]_{\circ_1, \circ_2}) \\ &= [x, y]_{\circ_2} \circ_1 z - z \circ_2 [x, y]_{\circ_1}, \end{aligned}$$

$$(22) \quad \begin{aligned} & [[x, y]_{\circ_2, \circ_1}, z]_{\circ_2, \circ_1} - [x, [y, z]_{\circ_2, \circ_1}]_{\circ_2, \circ_1} - ([y, x]_{\circ_2, \circ_1}, z]_{\circ_2, \circ_1} - [y, [x, z]_{\circ_2, \circ_1}]_{\circ_2, \circ_1}) \\ &= [x, y]_{\circ_1} \circ_2 z - z \circ_1 [x, y]_{\circ_2}, \quad \forall x, y, z \in A. \end{aligned}$$

Proof. For $x, y, z \in A$, we have

$$\begin{aligned} & [[x, y]_{\circ_1, \circ_2}, z]_{\circ_1, \circ_2} - [x, [y, z]_{\circ_1, \circ_2}]_{\circ_1, \circ_2} = x \circ_1 (z \circ_2 y) + (y \circ_1 z) \circ_2 x - z \circ_2 (x \circ_1 y) - (y \circ_2 x) \circ_1 z, \\ & [[y, x]_{\circ_1, \circ_2}, z]_{\circ_1, \circ_2} - [y, [x, z]_{\circ_1, \circ_2}]_{\circ_1, \circ_2} = (x \circ_1 z) \circ_2 y + y \circ_1 (z \circ_2 x) - z \circ_2 (y \circ_1 x) - (x \circ_2 y) \circ_1 z. \end{aligned}$$

This proves Eq. (21). Since \circ_1 and \circ_2 are symmetric in the definition of matching dialgebras, Eq. (22) also follows. \square

Definition 2.14. ([5, 6, 19]) A **pre-Lie algebra** is a pair (A, \circ) where A is a \mathbf{k} -vector space and \circ is a binary operation on A that satisfies

$$(23) \quad (x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z), \forall x, y, z \in A.$$

If in addition, (A, \circ) satisfies

$$(24) \quad (x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y), \forall x, y, z \in A,$$

then (A, \circ) is called a **bi-symmetric algebra** or an **assosymmetric algebra**.

Proposition 2.15. Let (A, \circ_1, \circ_2) be a matching dialgebra with both \circ_1 and \circ_2 commutative. With the notations in Eq. (18), $(A, [\ , \]_{\circ_1, \circ_2})$ is an assosymmetric algebra.

Proof. By Lemma 2.13, $(A, [\ , \]_{\circ_1, \circ_2})$ and $(A, [\ , \]_{\circ_2, \circ_1})$ are pre-Lie algebras. By [2], a pre-Lie algebra is an assosymmetric algebra if and only if its opposite algebra is also a pre-Lie algebra. Since $(A, -[\ , \]_{\circ_2, \circ_1})$ is the opposite algebra of $(A, [\ , \]_{\circ_1, \circ_2})$ by Remark 2.10, the conclusion follows. \square

Next we consider the case when \circ_1 and \circ_2 might not be commutative.

Definition 2.16. ([18]) A **left PostLie algebra** is a triple $(A, \circ, [\ , \])$ consisting of a vector space A , a product \circ and a skew-symmetric product $[\ , \]$ satisfying the relations:

$$(25) \quad [[x, y], z] + [[z, x], y] + [[y, z], x] = 0,$$

$$(26) \quad (x \circ y) \circ z - x \circ (y \circ z) - (y \circ x) \circ z + y \circ (x \circ z) - [x, y] \circ z = 0,$$

$$(27) \quad z \circ [x, y] - [z \circ x, y] - [x, z \circ y] = 0.$$

Proposition 2.17. Let (A, \circ_1, \circ_2) be a matching dialgebra such that $[x, y]_{\circ_1} = [x, y]_{\circ_2}$. Then $(A, [\ , \]_{\circ_1, \circ_2}, [\ , \]_{\circ_1})$ is a left PostLie algebra.

Proof. Eq. (25) follows from the fact that $(A, [\ , \]_{\circ_1})$ is a Lie algebra. Since $[x, y]_{\circ_1} = [x, y]_{\circ_2}$, by Eq. (21) we have

$$\begin{aligned} & [[x, y]_{\circ_1, \circ_2}, z]_{\circ_1, \circ_2} - [x, [y, z]_{\circ_1, \circ_2}]_{\circ_1, \circ_2} - ([y, [x, z]_{\circ_1, \circ_2}]_{\circ_1, \circ_2} - [y, [x, z]_{\circ_1, \circ_2}]_{\circ_1, \circ_2}) \\ &= [[x, y]_{\circ_1}, z]_{\circ_1, \circ_2}, \quad \forall x, y, z \in A. \end{aligned}$$

Thus Eq (26) holds.

Further by definition,

$$\begin{aligned} & [z, [x, y]_{\circ_1}]_{\circ_1, \circ_2} - [[z, x]_{\circ_1, \circ_2}, y]_{\circ_1} - [x, [z, y]_{\circ_1, \circ_2}]_{\circ_1} \\ &= y \circ_1(z \circ_1 x) - y \circ_2(z \circ_1 x) - x \circ_1(z \circ_1 y) + x \circ_2(z \circ_1 y) \\ &= y \circ_1(z \circ_1 x) - ((z \circ_1 x) \circ_2 y + y \circ_1(z \circ_1 x) - (z \circ_1 x) \circ_1 y) \\ &\quad - x \circ_1(z \circ_1 y) + ((z \circ_1 y) \circ_2 x + x \circ_1(z \circ_1 y) - (z \circ_1 y) \circ_1 x) \\ &= -z \circ_1(x \circ_2 y) + z \circ_1(x \circ_1 y) - z \circ_1(y \circ_1 x) + z \circ_1(y \circ_2 x) \\ &= z \circ_1(-x \circ_2 y + x \circ_1 y - y \circ_1 x + y \circ_2 x) = 0. \end{aligned}$$

Therefore Eq. (27) holds. \square

3. KOSZULITY, FREE OBJECTS AND OPERADIC HOMOLOGY OF THE MATCHING DIALGEBRAS

We apply the rewriting method to prove that the operad of matching dialgebras is Koszul in Section 3.1 and to construct free matching dialgebras in Section 3.2. We finally give the operadic homology of the matching dialgebras in Section 3.3.

3.1. The operad $As^{(2)}$ of matching dialgebras is Koszul. We recall the following general result [10, 12] on nonsymmetric (ns) operads.

Theorem 3.1. (*Rewriting method for ns operads [10, 12]*). *Let $\mathcal{P} := \mathcal{P}(E; R)$ be a reduced quadratic ns operad. If its generating space E admits an ordered basis for which there exists a suitable order on planar trees such that every critical monomial is confluent, then the ns operad \mathcal{P} is Koszul.*

Change of notations: To avoid confusion with the conventional notations in an operad, in the rest of the paper, we will use the notations $\mu_1 := \circ_1$ and $\mu_2 := \circ_2$, and reserve $\circ_1 := ((\cdot, \cdot), \cdot)$ and $\circ_2 := (\cdot, (\cdot, \cdot))$ for the compositions in the operad $As^{(2)}$. Thus $f \circ_i g$ is the composition of g on the i -th input of f , $i = 1, 2$.

Theorem 3.2. *The operad $As^{(2)}$ of matching dialgebras is a Koszul operad.*

Proof. The generating space E of binary operations for the operad $As^{(2)}$ has an ordered basis $\{\mu_1 > \mu_2\}$ corresponding to the two binary operations in the matching dialgebra.

With the notations just fixed above, the space R of relations of the operad $As^{(2)}$ is spanned by the following set of relators coming from the matching dialgebra in Definition 2.1:

$$(28) \quad \mu_i \circ_1 \mu_j - \mu_i \circ_2 \mu_j = 0, \quad i, j \in \{1, 2\},$$

with the first terms as the leading terms. These relations define the rewriting rule

$$\mu_i \circ_1 \mu_j \mapsto \mu_i \circ_2 \mu_j, \quad i, j \in \{1, 2\},$$

which gives rise to the critical monomials

$$(29) \quad \mu_i \circ_1 \mu_j \circ_1 \mu_k, \quad i, j, k \in \{1, 2\}.$$

By Theorem 3.1, it is enough to check that these critical monomials are confluent. Since the relations in Eq. (28) are of associative type, the confluence of the critical monomials can be checked by their diamonds in the pentagon shown in Figure 3.1 in page 8. More precisely, for the critical monomial $\mu_i \circ_1 \mu_j \circ_1 \mu_k$, the left side of the pentagon is

$$\mu_i \circ_1 (\mu_j \circ_1 \mu_k) = \mu_i \circ_1 (\mu_j \circ_2 \mu_k) = (\mu_i \circ_1 \mu_j) \circ_2 \mu_k = (\mu_i \circ_2 \mu_j) \circ_2 \mu_k = \mu_i \circ_2 (\mu_j \circ_1 \mu_k) = \mu_i \circ_2 (\mu_j \circ_2 \mu_k),$$

and the right side of the pentagon is

$$(\mu_i \circ_1 \mu_j) \circ_1 \mu_k = (\mu_i \circ_2 \mu_j) \circ_1 \mu_k = \mu_i \circ_2 (\mu_j \circ_2 \mu_k).$$

Hence the critical monomial is confluent, as needed. \square

3.2. Free matching dialgebras. The concept of a free matching dialgebra is defined by the usual universal property coming from the forgetful functor from the category of matching dialgebras to the category of sets. We first give in Section 3.2.1 a construction of the free matching dialgebra on a vector space by the rewriting method in Section 3.1 which also gives the graded vector space of the operad of matching dialgebras. We then give in Section 3.2.2 constructions of the free matching dialgebras from tensor algebras.

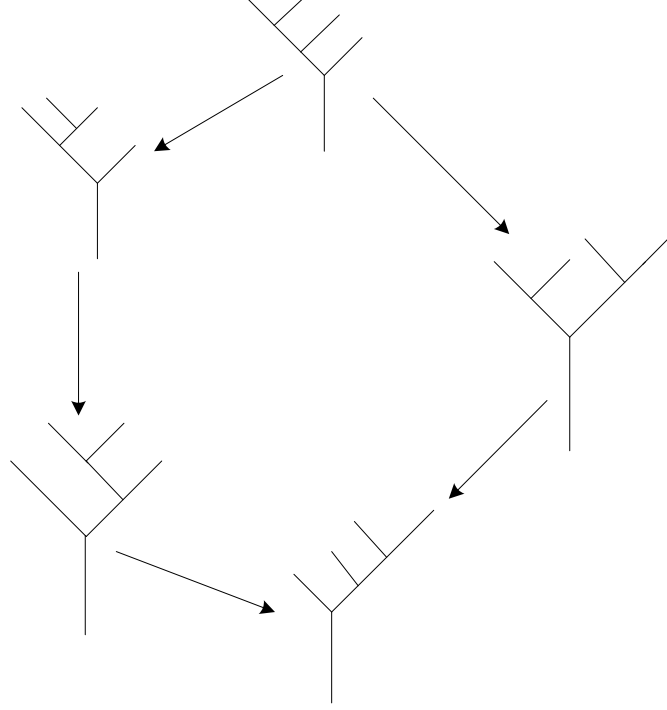


Figure 3.1

3.2.1. Free matching matching dialgebras.

Theorem 3.3. *Let V be a vector space with a basis X and let $\{\bullet_1, \bullet_2\}$ be a set. Define the disjoint union*

$$\begin{aligned} \text{MDS}(X) : &= X \coprod \left(\coprod_{k \geq 1} X \times (\{\bullet_1, \bullet_2\} \times X)^k \right) \\ &= \left\{ u := x_1 \bullet_{i_1} x_2 \bullet_{i_2} \cdots \bullet_{i_{k-1}} x_k \mid x_i \in X, 1 \leq i \leq k, i_j \in \{1, 2\}, 1 \leq j \leq k-1, k \geq 1 \right\} \end{aligned}$$

with the convention that $u = x_1$ when $k = 1$. Let

$$\text{MDA}(V) := \mathbf{k}\text{MDS}(X)$$

be the \mathbf{k} -linear span. For $u = x_1 \bullet_{i_1} \cdots \bullet_{i_{k-1}} x_k$ and $v = x_{k+1} \bullet_{i_{k+1}} \cdots \bullet_{i_{k+\ell-1}} x_{k+\ell}$ in $\text{MDS}(X)$, define

$$(30) \quad \mu_1(u, v) : = x_1 \bullet_{i_1} \cdots \bullet_{i_{k-1}} x_k \bullet_1 x_{k+1} \bullet_{i_{k+1}} \cdots \bullet_{i_{k+\ell-1}} x_{k+\ell},$$

$$(31) \quad \mu_2(u, v) : = x_1 \bullet_{i_1} \cdots \bullet_{i_{k-1}} x_k \bullet_2 x_{k+1} \bullet_{i_{k+1}} \cdots \bullet_{i_{k+\ell-1}} x_{k+\ell}.$$

Extend μ_1 and μ_2 to $\text{MDA}(V)$ by bilinearity. Then the triple $(\text{MDA}(V), \mu_1, \mu_2)$, together with the natural embedding

$$j_V : V \rightarrow \text{MDA}(V), \quad v \mapsto v, \quad v \in V,$$

is the free matching dialgebra on V . More precisely, the following universal property is satisfied: Let (R, μ'_1, μ'_2) be a matching dialgebra with operations μ'_1 and μ'_2 . Let $f : V \rightarrow R$ be a linear map. Then there exists a unique homomorphism

$$\bar{f} : \text{MDA}(V) \rightarrow R$$

of matching dialgebras such that $f = \bar{f} \circ j_V$.

Proof. The rewriting system of the ns operad $As^{(2)}$ is shown to be convergent in the proof of Theorem 3.2. Thus, as in [12, Section 8.1], the operad admits a PBW basis consisting of the right combs shown in Figure 3.2.1 in page 9 with vertices decorated by μ_1 and μ_2 . Thus $As_n^{(2)}$ has a \mathbf{k} -

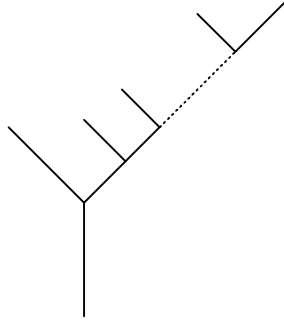


Figure 3.2.1

basis $\{(\mu_{i_1}, \dots, \mu_{i_{n-1}}) \mid i_j \in \{1, 2\}, 1 \leq j \leq n-1\}$. Since for a vector space V , the free $As^{(2)}$ -algebra on V is $As^{(2)}(V) = \bigoplus_{n \geq 1} As_n^{(2)} \otimes V^{\otimes n}$, we see that the free matching dialgebra on V has a basis given in the theorem. The given multiplications can then be derived from the compositions of the operations of the operad $As^{(2)}$. \square

3.2.2. Free matching dialgebras in terms of tensor algebras. We now give two more constructions of free matching dialgebras from tensor algebras.

For a vector space W , let $\bar{T}_*(W) = \bigoplus_{k \geq 1} W^{\otimes k}$ denote the non-unitary tensor algebra where the (tensor) product is denoted by \otimes_* .

Theorem 3.4. *Let V be a vector space over \mathbf{k} .*

- (a) *Equip $\bar{T}_{*1}(\bar{T}_{*2}(V))$ with products μ_1 and μ_2 as follows. First define μ_1 to be \otimes_{*1} . Next for $u = u_1 \otimes_{*1} \dots \otimes_{*1} u_m$ and $v = v_1 \otimes_{*1} \dots \otimes_{*1} v_n$ in $\bar{T}_{*1}(\bar{T}_{*2}(V))$ with $u_i, v_j \in \bar{T}_{*2}(V)$ for $1 \leq i \leq m, 1 \leq j \leq n$, define*

$$(32) \quad \mu_2(u, v) := u_1 \otimes_{*1} \dots \otimes_{*1} (u_m \otimes_{*2} v_1) \otimes_{*1} \dots \otimes_{*1} v_n.$$

*Then $(\bar{T}_{*1}(\bar{T}_{*2}(V)), \mu_1, \mu_2)$ is the free matching dialgebra on V .*

- (b) *Similarly, $\bar{T}_{*1}(\bar{T}_{*2}(V))$ with products μ_2 and μ_1 defined as follows is the free matching dialgebra on V . First, define μ_2 to be \otimes_{*2} . Next for $u = u_1 \otimes_{*2} \dots \otimes_{*2} u_m$ and $v = v_1 \otimes_{*2} \dots \otimes_{*2} v_n$ in $\bar{T}_{*1}(\bar{T}_{*2}(V))$ with $u_i, v_j \in \bar{T}_{*1}(V)$ for $1 \leq i \leq m, 1 \leq j \leq n$, define*

$$(33) \quad \mu_1(u, v) := u_1 \otimes_{*2} \dots \otimes_{*2} (u_m \otimes_{*1} v_1) \otimes_{*2} \dots \otimes_{*2} v_n.$$

Proof. (a). We only need to show that $\bar{T}_{*1}(\bar{T}_{*2}(V)) \cong \text{MDA}(V)$ as matching dialgebras, where $\text{MDA}(V)$ is the free matching dialgebra defined in Theorem 3.3.

Construct homomorphisms $\varphi : \bar{T}_{*1}(\bar{T}_{*2}(V)) \rightarrow \text{MDA}(V)$ and $\psi : \text{MDA}(V) \rightarrow \bar{T}_{*1}(\bar{T}_{*2}(V))$ as follows. Let

$$(34) \quad u = u_1 \otimes_{*1} \dots \otimes_{*1} u_m \in \bar{T}_{*1}(\bar{T}_{*2}(V)) \text{ with } u_i = u_{i1} \otimes_{*2} \dots \otimes_{*2} u_{in_i} \in \bar{T}_{*2}(V), 1 \leq i \leq m.$$

Define

$$\varphi(u) := (u_{11} \bullet_2 \cdots \bullet_2 u_{1n_1}) \bullet_1 \cdots \bullet_1 (u_{m1} \bullet_2 \cdots \bullet_2 u_{mn_m}).$$

Conversely, suppose $v = v_1 \bullet_{i_1} \cdots \bullet_{i_{k-1}} v_k$ is in $\text{MDA}(V)$ with $i_j \in \{1, 2\}$, $1 \leq j \leq k-1$, then there are $1 = p_1 < \cdots < p_\ell \leq k$ and $k_j \geq 0$, $1 \leq j \leq \ell$ such that:

$$(35) \quad v = \bar{v}_1 \bullet_1 \cdots \bullet_1 \bar{v}_\ell,$$

where $\bar{v}_j = v_{p_j} \bullet_2 \cdots \bullet_2 v_{p_j+k_j}$. So in particular $p_{j+1} = p_j + k_j + 1$, $1 \leq j \leq \ell - 1$. Then define

$$\psi(v) := \bar{u}_1 \otimes_{*1} \cdots \otimes_{*1} \bar{u}_\ell,$$

where $\bar{u}_j = x_{p_j} \otimes_{*2} \cdots \otimes_{*2} x_{p_j+k_j}$, $1 \leq j \leq \ell$.

We next check that φ and ψ are inverse of each other. For u in $\bar{T}_{*1}(\bar{T}_{*2}(V))$ defined in Eq. (34) and v in $\text{MDA}(V)$ defined in Eq. (35), we have

$$\begin{aligned} (\psi \circ \varphi)(u) &= \psi(u_{11} \otimes_{o_2} \cdots \otimes_{o_2} u_{1n_1} \otimes_{o_1} \cdots \otimes_{o_1} u_{m1} \otimes_{o_2} \cdots \otimes_{o_2} u_{mn_m}) \\ &= (u_{11} \bullet_2 \cdots \bullet_2 u_{1n_1}) \bullet_1 \cdots \bullet_1 (u_{m1} \bullet_2 \cdots \bullet_2 u_{mn_m}) \\ &= u. \end{aligned}$$

Similarly $\varphi \circ \psi = \text{id}_{\text{MDA}(V)}$. □

3.3. Operadic homology of matching dialgebras. In Theorem 3.2, we proved that the operad $As^{(2)}$ of matching dialgebras is Koszul. On the other hand, we know [12] that, for a Koszul operad, the Quillen homology is equal to the homology of the bar construction defined in [12] of the algebra over an operad. So the Quillen homology of the matching dialgebra is the desired operadic homology and, in order to give the operadic homology of matching dialgebra, we only need to give the construction of the Koszul dual cooperad $As^{(2)\dagger}$.

Proposition 3.5. *The operad $As^{(2)}$ is a Koszul self dual operad. More generally, for $k \geq 2$, let $As^{(k)}$ denote the binary quadratic ns operad whose space of binary operations has a basis μ_i , $1 \leq i \leq k$ and whose space R of relations is spanned by*

$$\mu_i \circ_1 \mu_j - \mu_i \circ_2 \mu_j, \quad 1 \leq i, j \leq k.$$

Here we use the notation $\circ_1 := ((\cdot, \cdot), \cdot)$, $\circ_2 := (\cdot, (\cdot, \cdot))$ before Theorem 3.2. Then $As^{(k)}$ is a Koszul self dual operad.

Proof. In the Koszul dual operad $As^{(k)\dagger}$ of $As^{(k)}$, let μ_i^* , $1 \leq i \leq k$, be the dual basis of μ_i , $1 \leq i \leq k$. From [7] and [12, Chapter 7], we find that the space of relations of $As^{(k)\dagger}$, being the annihilator of R , is precisely the subspace with a basis

$$\{\mu_i^* \circ_1 \mu_j^* - \mu_i^* \circ_2 \mu_j^* \mid 1 \leq i, j \leq k\}.$$

This proves the lemma. □

By the above fact and Theorem 3.2, we can compute the Koszul complex for matching dialgebras.

Corollary 3.6. *Let A be a matching dialgebra, i.e. a $As^{(2)}$ -algebra. The Koszul complex of A is given by*

$$\mathcal{C}_\bullet^{As^{(2)}}(A) := (As^{(2)\dagger}(A), d).$$

More precisely, $\mathcal{C}_{n-1}^{As^{(2)}}(A) = As_n^{(2)i} \otimes A^{\otimes n}$ with d satisfying

$$\begin{aligned} d(\lambda \otimes (a_1, \dots, a_n)) &:= \sum_{i=1}^{n-1} (-1)^{i+1} d_i(\lambda \otimes (a_1, \dots, a_n)) \\ &:= \sum_{i=1}^{n-1} (-1)^{i+1} \xi \otimes (a_1, \dots, a_{i-1}, \mu(a_i, a_{i+1}), \dots, a_n) \end{aligned}$$

for

$$\Delta_{(1)}(\lambda) = \sum_{i=1}^{n-1} (\xi; id, \dots, id, \mu, id, \dots, id)$$

where $\mu \in E = \mathbf{k}\{\mu_1, \mu_2\}$, $\lambda \in As_n^{(2)i}$, $\xi \in As_{n-1}^{(2)i}$ and $a_j \in A$, $1 \leq j \leq n$ and $\Delta_{(1)}$ is the infinitesimal decomposition map in cooperad $As^{(2)i}$.

Proof. Koszul complexes of symmetric operads are considered in [12, Chapter 12.1.2]. Since the operad $As^{(2)}$ is an ns operad, we give a direct proof.

Recall from Section [12, Chapter 11.2.2] that the differential map d is the unique coderivation which extends the composite of the morphism κ defined in [12, Chapter 7.4.1] with the product γ_A given in [12, Chapter 5.2.3] of A . It is explicitly given by the following composite

$$As^{(2)i}(A) \xrightarrow{\Delta_{(1)} \circ Id_A} (As^{(2)i} \circ_{(1)} As^{(2)i}) \circ A \xrightarrow{(Id \circ_{(1)} \kappa) \circ Id_A} (As^{(2)i} \circ_{(1)} As^{(2)}) \circ A \xrightarrow{\gamma_A} As^{(2)i} \circ As^{(2)} \circ A \xrightarrow{Id \circ \gamma_A} As^{(2)i}(A).$$

Since the operad $As^{(2)}$ is binary, the infinitesimal decomposition map $\Delta_{(1)}$ splits the n -ary co-operation λ of $As^{(2)i}$ into an $(n-1)$ -ary co-operation and a binary co-operation of $As^{(2)i}$. By the morphism $\kappa : As^{(2)i} \rightarrow As^{(2)}$, the latter co-operation is viewed as a binary operation μ of the operad $As^{(2)}$. The sum is over all these possibilities of splitting.

Thus it is enough to check $d^2 = 0$. Since λ is fixed, we just replace $d(\lambda \otimes (a_1, \dots, a_n))$ by $d(a_1, \dots, a_n)$ to simplify the notation. By the definition of d given in Corollary 3.6, we have

$$\begin{aligned} d^2(a_1, \dots, a_n) &= \sum_{i=1}^{n-1} (-1)^{i+1} \left(\sum_{j=1}^{n-2} (-1)^{j+1} (a_1, \dots, a_{j-1}, \nu(a_j, a_{j+1}), \dots, a_{i-1}, \mu(a_i, a_{i+1}), \dots, a_n) \right) \\ (36) \quad &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} c_{ij} \end{aligned}$$

with $\nu, \mu \in \{\circ_1, \circ_2\}$ and

$$c_{ij} = \begin{cases} 0 & i = j, \\ (-1)^{i+j+1} (a_1, \dots, a_{i-1}, \nu(\mu(a_i, a_{i+1}), a_{i+2}), \dots, a_n) & j = i+1, \\ (-1)^{i+j+1} (a_1, \dots, a_{i-1}, \mu(a_i, a_{i+1}), \dots, a_{j-1}, \nu(a_j, a_{j+1}), \dots, a_n) & j \geq i+2, \\ (-1)^{i+j} (a_1, \dots, a_{i-2}, \nu(a_{i-1}, \mu(a_i, a_{i+1})), \dots, a_n) & i = j+1, \\ (-1)^{i+j} (a_1, \dots, a_{j-1}, \nu(a_j, a_{j+1}), \dots, a_{i-1}, \mu(a_i, a_{i+1}), \dots, a_n) & i \geq j+2. \end{cases}$$

If $j = i+1$, we have

$$c_{ij} = (-1)^{i+j+1} (a_1, \dots, a_{i-1}, \nu(\mu(a_i, a_{i+1}), a_{i+2}), \dots, a_n)$$

and

$$\begin{aligned}
c_{ji} &= (-1)^{i+j}(a_1, \dots, a_{j-2}, \nu(a_{j-1}, \mu(a_j, a_{j+1})), \dots, a_n) \\
&= (-1)^{i+j}(a_1, \dots, a_{i-1}, \nu(a_i, \mu(a_{i+1}, a_{i+2})), \dots, a_n) \\
&= -(-1)^{i+j+1}(a_1, \dots, a_{i-1}, \nu(\mu(a_i, a_{i+1}), a_{i+2})) \\
&= -c_{ij}.
\end{aligned}$$

If $j \geq i + 2$, we also have

$$c_{ji} = (-1)^{i+j}(a_1, \dots, a_{i-1}, \nu(a_i, a_{i+1}), \dots, a_{j-1}, \mu(a_j, a_{j+1}), \dots, a_n) = -c_{ij}.$$

Thus we have $c_{ji} = -c_{ij}$ for all $1 \leq i, j \leq n - 1$ and hence $d^2 = 0$. \square

Corollary 3.7. *Let \mathbf{k} be a field of characteristic zero. The operadic homology of the free $As^{(2)}$ -algebra $\bar{T}_{*2}(\bar{T}_{*1}(V))$ given in Theorem 3.4 is equal to*

$$H_n^{As^{(2)}}(\bar{T}_{*2}(\bar{T}_{*1}(V))) := \begin{cases} V, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1. \end{cases}$$

Proof. By Theorem 3.2, $As^{(2)}$ is Koszul. Then the corollary is a direct consequence of [12, Theorem 12.1.6] since $\bar{T}_{*2}(\bar{T}_{*1}(V))$ is the free $As^{(2)}$ -algebra on V by Theorem 3.4. \square

Acknowledgements. C. Bai would like to thank the support by NSFC (10920161) and SRFDP (200800550015). L. Guo thanks NSF grant DMS-1001855 for support. The authors thank J.-L. Loday, to whose memory this paper is dedicated to, and the editor for very helpful suggestions and comments.

REFERENCES

- [1] A. Andrada and S. Salamon, Complex product structure on Lie algebras, *Forum Math.* **17** (2005) 261-295.
- [2] C. Bai and D. J. Meng, On bi-symmetric algebras, *Bull. Inst. Math. Acad. Sinica* **26** (1998) 127-145.
- [3] C. Bai, Double constructions of Frobenius algebras, Connes cocycles and their duality, *J. Noncommu. Geom.* **4** (2010) 475-530.
- [4] C. Bai, L. Guo and X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, *Commun. Math. Phys.* **297** (2010) 553-596.
- [5] D. Burde, Left-symmetric algebras, or pre-Lie algebras in geometry and physics, *Cent. Eur. J. Math.* **4** (2006) 323-357.
- [6] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. Math.* **78** (1963) 267-288.
- [7] V. Ginzburg and M. Kapranov, Koszul duality for operads, *Duke Math. J.* **76** (1994), 203-272.
- [8] I. Z. Golubchik and V.V. Sokolov, Compatible Lie brackets and integrable equations of the principal chiral model type, *Funct. Anal. Appl.* **36** (2002) 172-181.
- [9] I. Z. Golubchik and V.V. Sokolov, Compatible Lie brackets and the Yang-Baxter equation, *Theor. Math. Phys.* **146** (2006) 159-169.
- [10] E. Hoffbeck, A Poincaré-Birkhoff-Witt criterion for Koszul operads, *Manuscripta Math.* **131** (2010) 87C110.
- [11] B. A. Kupersmidt, Phase spaces of algebras, Mathematics Publications and Other Works (2010). http://trace.tennessee.edu/utk_mathpubs/2.
- [12] J. -L. Loday and B. Vallette, Algebraic Operads, Grundlehren Math. Wiss. 346, Springer, Heidelberg, 2012.
- [13] S. Majid, Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations, *Pacific J. Math.* **141** (1990) 311-332.
- [14] A. V. Odesskii and V.V. Sokolov, Algebraic structures connected with pairs of compatible associative algebras, *Int. Math. Res. Notices* (2006) doi: 10.1155/IMRN/2006/43734.

- [15] A. V. Odesskii and V.V. Sokolov, Integrable matrix equations related to pairs of compatible associative algebras, *J. Phys. A: Math. Gen.* **39** (2006) (40) 12447-12456.
- [16] A. V. Odesskii and V.V. Sokolov, Compatible Lie brackets related to elliptic curve, *J. Math. Phys.* **47** (2006) 013506.
- [17] A. V. Odesskii and V.V. Sokolov, Pairs of compatible associative algebras, classical Yang-Baxter equation and quiver representations, *Commun. Math. Phys.* **278** (2008) 83-99.
- [18] B. Vallette, Homology of generalized partition posets, *J. Pure Appl. Algebra* (2007) 699-725.
- [19] E. B. Vinberg, Convex homogeneous cones, *Transl. of Moscow Math. Soc.* **12** (1963) 340-403.
- [20] G. W. Zinbiel, Encyclopedia of types of algebras 2010, in “Operads and Universal Algebra”, *Nankai Series in Pure, Applied Mathematics and Theoretical Physics* **9**, World Scientific, Singapore, 2012, 217-298.

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU, ZHEJIANG 310027, CHINA
E-mail address: tangmeng@zju.edu.cn

CHERN INSTITUTE OF MATHEMATICS AND LPMC, NANKAI UNIVERSITY, TIANJIN 300071, CHINA
E-mail address: baicm@nankai.edu.cn

SCHOOL OF MATHEMATICS AND STATISTICS, LANZHOU UNIVERSITY, LANZHOU, GANSU 730000, CHINA AND DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, RUTGERS UNIVERSITY, NEWARK, NJ 07102, USA
E-mail address: liguo@rutgers.edu